A higher dimensional generalization of Uchida's Hom conjecture

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Abstract

In the present paper, we prove that any open homomorphism between the absolute Galois groups of finitely generated fields of characteristic 0 arises from a unique embedding between the fields. This result may be regarded as a higher dimensional analogue of the Hom conjecture for number fields due to K. Uchida [whose proof was announced in a recent note due to OpenAI] and a complete generalization of the celebrated result due to F. Pop concerning the corresponding Isom-version. Since the original conjecture is applied to prove the higher dimensional analogue in the present paper, we also discuss/review the announced proof of the conjecture in the note from the point of view of the author.

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Introduction

In the present paper, we discuss a higher dimensional analogue of K. Uchida's Hom conjecture for the absolute Galois groups of number fields, which may be regarded as an absolute anabelian Grothendieck conjecture-type question. In [23], K. Uchida asks

> *whether or not any open homomorphism between the absolute Galois groups [or more generally, solvably closed quotients of them] of number fields arise from field embeddings.*

While there were several significant progresses concerning this question due to K. Uchida himself and Y. Hoshi [cf., e.g., [10], [23]], this question has been a long-standing open question. However, very recently, a complete resolution of this long-standing open question was announced by OpenAI [cf. [16]]. It appears to the author that the note due to OpenAI contains an interesting new idea that enables us to drop an assumption [concerning the compatibility with cyclotomic characters] in the Y. Hoshi's result.

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One of our purpose of the present paper is to review and clarify the proof of them. Another purpose of the present paper is to prove a stronger higher dimensional analogue of the Uchida conjecture by reducing to the original Uchida conjecture and a S. Mochizuki's highly nontrivial result in relative anabelian geometry. [This reduction part was already obtained in October 2025 but the author could not give a proof of the original conjecture.]

In order to state our main result precisely, we introduce some notations. For each field K, we shall write G_K for the absolute Galois group of K with respect to a fixed separable closure of K. For each pair of fields K_1, K_2, we shall write

$$\mathrm{Emb}(K_2, K_1)$$

for the set of embeddings $K_2 \hookrightarrow K_1$ of fields;

$$\mathrm{Hom}^{\mathrm{open}}(G_{K_1}, G_{K_2})/\mathrm{Inn}(G_{K_2})$$

for the set of open homomorphisms $G_{K_1} \to G_{K_2}$, considered up to compositions with inner automorphisms of G_{K_2}. Then our main result is the following [cf. Theorem 3.5]:

Theorem A. *Let K_1, K_2 be finitely generated fields of characteristic 0. Then the natural map*

$$\psi : \mathrm{Emb}(K_2, K_1) \longrightarrow \mathrm{Hom}^{\mathrm{open}}(G_{K_1}, G_{K_2})/\mathrm{Inn}(G_{K_2})$$

is bijective.

In the author's knowledge, the resolution of the Uchida's conjecture due to Y. Hoshi [partially] and OpenAI [completely by reducing to the Hoshi's result] may be regarded as the first unconditional result for absolute open homomorphisms in anabelian geometry. Furthermore, Theorem A may be regarded as a certain generalization of F. Pop's famous result for the corresponding Isom-version [cf., e.g., [17], [18]]. The key ingredients of the proof of Theorem A is

- the Uchida conjecture due to Y. Hoshi and OpenAI [cf. [10], [16]],
- the relative birational anabelian Grothendieck conjecture for arbitrary dimension due to S. Mochizuki [cf. [12]], and
- the semi-absoluteness of open homomorphisms [i.e., roughly speaking, the preservation of geometric fundamental groups] based on the technique of reconstructions of valuations due to H. Köenigsmann.

Let Σ be a nonempty set of prime numbers. Then, if we replace the Mochizuki's result by a geometrically pro-p relative birational anabelian result obtained by S. Corry and F. Pop [cf. [1]], then one may prove the geometrically pro-Σ version of Theorem A. [Note that the semi-absoluteness is much easier to prove in the case where Σ is not equal to the set of all prime numbers, and the result due to Corry and Pop may be regarded as an application of the Mochizuki's work.] On the other hand, it is natural to ask whether or not the positive characteristic analogue of Theorem A holds. However, at the time of writing of the present paper, the author has no clue to answer to this question [cf. Remark 3.5.2]. One may also ask whether or not the finite step solvable quotient analogue of Theorem A holds. [Note that the solvably closed case for number fields is discussed in the present paper.] However, since there exist numerous obstacles that the author has no clue to overcome, this question is also out of the scope of the present paper.

Moreover, one may obtain the following immediate consequence of Theorem A [cf. Corollary 3.6]:

Corollary B. *Any open homomorphism between the absolute Galois groups of finitely generated fields of characteristic 0 with same transcendental degree is injective. In particular, the absolute Galois groups of finitely generated fields of characteristic 0 satisfy the Hopfian property.*

Note that Corollary B may be regarded as an affirmative answer to the open question in [15], Question 12.3.3.

The present paper is organized as follows. In §1, by using the main idea of the note due to OpenAI, we review/discuss the proof of the Uchida conjecture in a somewhat different style that resorts to the Grunewald-Segal theorem [cf. [6], Theorem A]. In §2, by applying various algebraic number theoretic/arithmetic geometric results, we prove that any open homomorphism between the absolute Galois groups of finitely generated fields of characteristic 0 induces an open homomorphism between the absolute Galois groups of base number fields [i.e., so called "semi-absoluteness" in the language of [14]]. In §3, by applying a Mochizuki's famous result in relative anabelian geometry, together with the results obtained in §1, §2, we prove Theorem A. We also discuss a certain analogue of Theorem A for suitable algebraic varieties over finitely generated fields of characteristic 0 [cf. Theorem 3.7].

Notations and Conventions

Numbers: The notation $\mathfrak{Primes}$ will be used to denote the set of prime numbers. We shall refer to a finite extension field of the field of rational numbers $\mathbb{Q}$ as a *number field*. The notation $\widehat{\mathbb{Z}}$ will be used to denote the profinite completion of the additive group $\mathbb{Z}$ of integers. For each $l \in \mathfrak{Primes}$, the notation $\mathbb{Z}_l$ will be used to denote the maximal pro-l quotient of $\widehat{\mathbb{Z}}$. If p is a prime number, then $\mathbb{Q}_p$ will be used to denote the field of p-adic numbers.

Fields: Let F be a field of characteristic 0; n a positive integer; p a prime number. Then we shall write $\overline{F}$ for the algebraic closure [determined up to isomorphisms over F] of F; $G_F \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{F}/F)$;

$$\mu_n(F) \stackrel{\text{def}}{=} \{x \in F \mid x^n = 1\}; \quad \mu_{p^\infty}(F) \stackrel{\text{def}}{=} \bigcup_m \mu_{p^m}(F); \quad F_{p^\infty} \stackrel{\text{def}}{=} F(\mu_{p^\infty}(\overline{F})).$$

where m ranges over the positive integers. We shall fix a primitive n-th root of unity $\zeta_n \in \overline{F}$.

Profinite groups: Let G be a profinite group. For each nonempty set of prime numbers Σ, we shall write G^Σ for the maximal pro-Σ quotient of G. For each prime number l, we shall write $G^l \stackrel{\text{def}}{=} G^{\{l\}}$. We shall write G^{ab} for the quotient of G by $\overline{[G,G]} \subseteq G$, i.e., the maximal abelian quotient of G in the category of profinite groups. If G is abelian, then we shall write $G[p^\infty] \subseteq G$ for the subgroup of p-power torsion elements for each prime number p; $G_{\mathrm{tor}} \subseteq G$ for the subgroup of torsion elements. If $H \subseteq G$ is a closed subgroup, then we shall write $N_G(H)$ for the normalizer of H in G.

Fundamental groups: For a connected Noetherian scheme S, we shall write Π_S for the étale fundamental group of S, relative to a suitable choice of basepoint. Let K be a field of characteristic 0; X an algebraic variety [i.e., a scheme that is separated, of finite type, and geometrically integral] over K. Then we shall write $K(X)$ for the function field of X;

$$\Delta_X \stackrel{\text{def}}{=} \Pi_{X \times_K \overline{K}}; \quad \Delta_{K(X)} \stackrel{\text{def}}{=} G_{K(X) \otimes_K \overline{K}}.$$

In particular, we have natural exact sequences of profinite groups

$$1 \longrightarrow \Delta_X \longrightarrow \Pi_X \longrightarrow G_K \longrightarrow 1, \quad 1 \longrightarrow \Delta_{K(X)} \longrightarrow G_{K(X)} \longrightarrow G_K \longrightarrow 1$$

[cf. [5], Exposé IX, Théorème 6.1].

1 Proof of Uchida's conjecture

In the present section, by using the main idea of the discussion in the recent note due to OpenAI, we revisit the proof of the Uchida's conjecture. While our exposition is based on the key observation concerning *finiteness of fibers* in the note, the author chooses to apply the Grunewald-Segal theorem [that does not appear in the note] to complete the proof after the observation. It appears to the author that the Grunewald-Segal theorem is useful in the situation related to finite partitions of primes.

First, the following is a basic setup of Uchida's conjecture:

Definition 1.1. For each $i = 1, 2$, let K_i be a number field; Ω_i a solvably closed Galois extension field of K_i. For each $i = 1, 2$, write $G_i \stackrel{\text{def}}{=} \mathrm{Gal}(\Omega_i/K_i)$. Let

$$\phi : G_1 \longrightarrow G_2$$

be an open homomorphism.

In this setting, the following highly nontrivial result is proved by Hoshi [cf. [10], Theorem]:

Theorem 1.2. *In the notation of Definition 1.1, suppose that ϕ is compatible with the respective cyclotomic characters. Then ϕ arises from a uniquely determined commutative diagram*

$$\begin{array}{ccc} \Omega_2 & \longrightarrow & \Omega_1 \\ \uparrow & & \uparrow \\ K_2 & \longrightarrow & K_1 \end{array}$$

of field embeddings, where the vertical arrows denote the natural inclusions.

OpenAI proved that the above compatibility with the cyclotomic characters is automatic. To discuss this, we introduce some notations as follows:

Definition 1.3. Let K be a number field. Then we shall write $\mathbb{V}(K)$ for the set of finite primes on K; $\mathbb{V}(K)^{d=1} \subseteq \mathbb{V}(K)$ for the subset of finite primes of absolute degree 1. For each $v \in \mathbb{V}(K)$, write p_v for the residue characteristic of v. For each $v \in \mathbb{V}(K)$ [together with its lifting to $\overline{\mathbb{Q}}$], let $\mathrm{Fr}_v \in G_K$ be a lifting of the Frobenius associated to v [together with its lifting to $\overline{\mathbb{Q}}$]. If K is Galois over $\mathbb{Q}$, then for each $v \in \mathbb{V}(K)$ unramified over $\mathbb{Q}$, we shall write $\mathrm{Fr}_{v/p_v} \in \mathrm{Gal}(K/\mathbb{Q})$ for the Frobenius element associated to v.

Here, we recall that the following result due to Uchida concerning local decomposition subgroups played an essential role in the Hoshi's proof [which also play an essential role in the note by OpenAI].

Lemma 1.4. *In the notation of Definition 1.1, let l be a prime number. Then, if a [or, equivalently, any] Sylow-l subgroup I_l of an inertia subgroup of G_1 associated to $v_1 \in \mathbb{V}(K_1)$ such that $p_{v_1} \neq l$ has infinite image via ϕ, then I_l maps isomorphically to an open subgroup of a Sylow-l subgroup of an inertia subgroup of G_2 associated to a prime $v_2 \in \mathbb{V}(K_2)$ such that $p_{v_2} \neq l$ via ϕ. In this situation, a decomposition subgroup $D_{v_1} \subseteq G_1$ associated to v_1 maps into a decomposition subgroup $D_{v_2} \subseteq G_2$ associated to v_2 via ϕ. Moreover, there exists a finite subset $S_l \subseteq \mathbb{V}(K_2)$ that contains the finite primes over l satisfying the following condition:*

For any $v_2 \in \mathbb{V}(K_2) \setminus S_l$, there exists $v_1 \in \mathbb{V}(K_1)$ such that $p_{v_1} \neq l$, and a [or, equivalently, any] Sylow-l subgroup of an inertia subgroup of G_1 associated to v_1 maps isomorphically to an open subgroup of a Sylow-l subgroup of an inertia subgroup of G_2 associated to v_2 via ϕ.

Proof. Lemma 1.4 follows immediately from [23], Proposition 1. □

The following is a key finiteness result [due to M. Emsalem — cf. [2]], which may be regarded as an application of M. Waldschmidt's theorem on p-adic transcendental number theory [cf. [24]]:

Proposition 1.5. *Let l be a prime number; K a number field; $\phi : G_K \to \mathbb{Z}_l$ a continuous nontrivial homomorphism. Then the set $\{v \in \mathbb{V}(K)^{d=1} \mid \phi(\mathrm{Fr}_v) = 0\}$ is finite. [In light of the structure of a local decomposition group, we note that the condition that $\phi(\mathrm{Fr}_v) = 0$ is independent of our choice of Fr_v if $v \nmid l$.]*

Proof. Write

$$\lambda_\phi : K \otimes_{\mathbb{Q}} \mathbb{Q}_l \xrightarrow{\sim} \prod_{v \in \mathbb{V}(K), v|l} K_v \longrightarrow \mathbb{Q}_l$$

for the homomorphism induced by the assignment $(\log_v(a_v))_{v|l} \mapsto \sum_{v|l} \phi(r_v(a_v))$ for each $(a_v)_{v|l} \in \prod_{v|l} \mathcal{O}_v^\times$, where K_v denotes the completion of K at v; $\mathcal{O}_v^\times$ denotes the unit group of K_v; $\log_v : \mathcal{O}_v^\times \to K_v$ denotes the l-adic logarithm at v; $r_v : \mathcal{O}_v^\times \to G_K^{\mathrm{ab}}$ denotes the composite of the local reciprocity map $\mathcal{O}_v^\times \hookrightarrow G_{K_v}^{\mathrm{ab}}$ with the natural homomorphism $G_{K_v}^{\mathrm{ab}} \to G_K^{\mathrm{ab}}$. In light of global class field theory, together with the finiteness of the class number h_K of K, to verify that $\phi = 0$, it suffices to verify that $\lambda_\phi = 0$.

Suppose that $\{v \in \mathbb{V}(K)^{d=1} \mid \phi(\mathrm{Fr}_v) = 0\}$ is infinite. Write $d \stackrel{\mathrm{def}}{=} [K : \mathbb{Q}]$. Fix a subset $S \subseteq \{v \in \mathbb{V}(K)^{d=1} \mid \phi(\mathrm{Fr}_v) = 0,\ v \nmid l\}$ of cardinality $> d(d-1)$ such that the primes in S lie over distinct prime numbers. For each $v \in S$, there exists an element $b_v \in \mathcal{O}_K$ such that $b_v \mathcal{O}_K = \mathfrak{p}_v^{h_K}$, where $\mathcal{O}_K$ denotes the ring of integers of K; $\mathfrak{p}_v$ denotes the maximal ideal of $\mathcal{O}_K$ associated to v. Next, we consider the following commutative diagram of natural embeddings:

$$\begin{array}{ccc} K \otimes_{\mathbb{Q}} \mathbb{Q}_l & \xrightarrow{\sim} & \prod_{v \in \mathbb{V}(K), v|l} K_v \\ \uparrow & & \downarrow \\ K & \longrightarrow & \prod_{\sigma \in \mathrm{Hom}(K, \overline{\mathbb{Q}}_l)} \overline{\mathbb{Q}}_l. \end{array}$$

Since S consists of primes of absolute degree 1, it is immediate that for each $(a_\sigma) \in \mathbb{Z}^{\oplus d} \setminus \{0\}$, the $\#S$ elements $\{\prod_{\sigma \in \mathrm{Hom}(K, \overline{\mathbb{Q}}_l)} \sigma(b_w)^{a_\sigma})\}_{w \in S}$ are multiplicatively independent [up to multiplication by roots of unity]. Thus, in light of the commutativity of the above diagram, we conclude from [24], p.119, Théorème 2.1.p, together with our assumption that $\#S > d(d-1)$, that the $\#S$ elements $\{(\log_v(b_w))_{v|l}\}_{w \in S}$ generate $\prod_{v|l} K_v$ as $\mathbb{Q}_l$-vector space. On the other hand, since $K^\times$ vanishes in the idèle class group of K, and the inertia subgroups of G_K associated to primes that do not lie over l vanish via ϕ, it immediate that for each $w \in S$,

$$\lambda_\phi((\log_v(b_w))_{v|l}) \ = \ \sum_{v|l} \phi(r_v(b_w)) \ = \ -h_K \phi(\mathrm{Fr}_w) \ = \ 0.$$

Thus, we conclude that $\lambda_\phi = 0$, as desired. This completes the proof of Proposition 1.5. □

Remark 1.5.1. One may not prove an analogous statement of Proposition 1.5 for the primes of arbitrary absolute degree. Indeed, anti-cyclotomic extensions of imaginary quadratic fields give such counter-examples.

As an immediate consequence of Proposition 1.5, one may give a slightly generalized version of Proposition 1.5 as follows. This assertion [or more precisely, the argument of the proof] will be applied several times in the remainder of the present section.

Corollary 1.6. *Let l be a prime number; K a number field; $\phi : G_K \to \mathbb{Z}_l$ a continuous homomorphism such that $\mathrm{Cor}_{K/\mathbb{Q}}(\phi) \neq 0$, where $\mathrm{Cor}_{K/\mathbb{Q}}(\phi) : G_{\mathbb{Q}} \to \mathbb{Z}_l$ denotes the corestriction of ϕ. Then the set $\{v \in \mathbb{V}(K) \mid \phi(\mathrm{Fr}_v) = 0\}$ is finite.*

Proof. To verify Corollary 1.6, in light of the transitivity of corestriction, by replacing K by its Galois closure over $\mathbb{Q}$, we may assume without loss of generality that K is Galois over $\mathbb{Q}$. Suppose that $\{v \in \mathbb{V}(K) \mid \phi(\mathrm{Fr}_v) = 0\}$ is infinite. Write $q : G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(K/\mathbb{Q})$ for the natural quotient. Then since $\mathrm{Gal}(K/\mathbb{Q})$ is finite, there exists an element $\lambda \in \mathrm{Gal}(K/\mathbb{Q})$ such that the set $S \stackrel{\mathrm{def}}{=} \{v \in \mathbb{V}(K) \mid \phi(\mathrm{Fr}_v) = 0,\ q(\mathrm{Fr}_{p_v}) = \lambda\}$ is infinite. Write $\Lambda \subseteq \mathrm{Gal}(K/\mathbb{Q})$ for the subgroup generated by λ; $K^{\Lambda} \subseteq K$ for the subfield fixed by Λ. Then, for each $v \in S$, if we write $v_{\Lambda} \in \mathbb{V}(K^{\Lambda})$ for the finite prime determined by v, then it follows immediately from the various definitions involved that:

- The absolute degree of v_{Λ} is equal to 1.
- $\mathrm{Cor}_{K/K^{\Lambda}}(\phi)(\mathrm{Fr}_{v_{\Lambda}}) = \phi(\mathrm{Fr}_v) = 0$ [cf. the definition of corestriction].

Thus, since S is infinite, we conclude from Proposition 1.5 that $\mathrm{Cor}_{K/K^{\Lambda}}(\phi) = 0$, hence from the transitivity $\mathrm{Cor}_{K/\mathbb{Q}}(\phi) = \mathrm{Cor}_{K^{\Lambda}/\mathbb{Q}}(\mathrm{Cor}_{K/K^{\Lambda}}(\phi))$ that $\mathrm{Cor}_{K/\mathbb{Q}}(\phi) = 0$. This contradicts our assumption. This completes the proof of Corollary 1.6. □

The main idea of the OpenAI's note is to apply Proposition 1.5 or Corollary 1.6 to extract a meaningful information on the correspondence of local decomposition subgroups as in Lemma 1.4. As a first step, we prove a certain uniformity lemma concerning the minimal cardinality of S_l in Lemma 1.4 [cf. Lemma 1.8]

Lemma 1.7. *In the notation of Definition 1.1, suppose that $\Omega_1 = \overline{\mathbb{Q}}$. Then there exists a positive integer n satisfying the following condition:*

> *For each sufficiently large prime number $l \equiv 1 \pmod{n}$, there exists a finite Galois extension $K_1 \subseteq L$ such that the restriction of ϕ to G_L is compatible with the respective l-adic cyclotomic characters.*

Proof. First, by replacing K_1, K_2 by their respective suitable finite extension fields, we may assume that K_1 is Galois over $\mathbb{Q}$, and ϕ is surjective. Write $n \stackrel{\mathrm{def}}{=} \#\mathrm{Gal}(K_1/\mathbb{Q})$. Let l be an odd prime number $\equiv 1 \pmod{n}$ such that for each positive integer a, if we write $E_a \stackrel{\mathrm{def}}{=} K_2(\mu_{l^a}(\Omega_2), l^{\frac{1}{l^a}})$, then there exists an isomorphism $\mathrm{Gal}(E_a/K_2) \stackrel{\sim}{\to} \mathbb{Z}/l^a\mathbb{Z} \rtimes (\mathbb{Z}/l^a\mathbb{Z})^{\times}$ [where we may assume that the semi-direct structure is consistent with varying a] that maps $\mathrm{Gal}(E_a/K_2(\mu_{l^a}(\Omega_2)))$ isomorphically onto $\mathbb{Z}/l^a\mathbb{Z}$. [Note that this condition is satisfied for sufficiently large prime numbers $l \equiv 1 \pmod{n}$.] We identify $\mathrm{Gal}(E_a/K_2)$ with $V \rtimes C \stackrel{\mathrm{def}}{=} \mathbb{Z}/l^a\mathbb{Z} \rtimes (\mathbb{Z}/l^a\mathbb{Z})^{\times}$ via the above isomorphism. Write $m \stackrel{\mathrm{def}}{=} \#C$; $M_a \subseteq E_a$ for the fixed subfield by the image of the splitting $C \hookrightarrow V \rtimes C = \mathrm{Gal}(E_a/K_2)$; $K_1 \subseteq L_1$ $(\subseteq \overline{\mathbb{Q}})$ for the finite Galois extension that corresponds to $\phi^{-1}(\mathrm{Gal}(\Omega_2/E_a)) \subseteq G_1$. Write L $(\subseteq \overline{\mathbb{Q}})$ for the Galois closure of $L_1(\zeta_l)$ over $\mathbb{Q}$. In particular, it holds that $L_1(\zeta_l) \subseteq L$, and $G_L \subseteq \phi^{-1}(\mathrm{Gal}(\Omega_2/K_2(\zeta_l)))$.

Next, let $x \in \mathrm{Ker}(\mathrm{Cor}_{M_a/M_{a-1}} : \mathrm{Hom}(G_{M_a}, \mathbb{Q}_l) \to \mathrm{Hom}(G_{M_{a-1}}, \mathbb{Q}_l))$ be a nonzero element [whose existence follows immediately from [15], Corollary 10.3.7]. Write $y_2 \stackrel{\mathrm{def}}{=} \mathrm{Res}_{E_a/M_a} x \in \mathrm{Hom}(G_{E_a}, \mathbb{Q}_l)$;

$W_2 \stackrel{\text{def}}{=} \mathbb{Q}_l[V]y_2 \subseteq \mathrm{Hom}(G_{E_a}, \mathbb{Q}_l)$, where we regard $\mathrm{Hom}(G_{E_a}, \mathbb{Q}_l)$ as a $\mathrm{Gal}(E_a/K_2)$-module in a natural way; $W \subseteq \mathrm{Hom}(G_L, \mathbb{Q}_l)$ for the image of W_2 via the natural injective homomorphism $\mathrm{Hom}(G_{E_a}, \mathbb{Q}_l) \hookrightarrow \mathrm{Hom}(G_L, \mathbb{Q}_l)$ induced by ϕ. Here, we note that C fixes y_2, hence that C acts on W_2. Moreover, in light of the choice of x, one may observe that there exist isomorphisms $(W \stackrel{\sim}{\to}) W_2 \stackrel{\sim}{\to} \mathbb{Q}_l(\zeta_{l^a})$ [that maps y_2 to 1], $C \stackrel{\sim}{\to} \mathrm{Gal}(\mathbb{Q}_l(\zeta_{l^a})/\mathbb{Q}_l)$ compatible with the respective natural actions. In light of Chebotarev's density theorem, fix an infinite subset $I \subseteq \mathbb{V}(K_2)^{d=1} \setminus S_l$ such that the image of Fr_w via the quotient $G_{K_2} \twoheadrightarrow \mathrm{Gal}(E_a/K_2)$ coincides with a generator of C for each $w \in I$. Then it follows immediately from the definition of S_l, together with the finiteness of $\mathrm{Gal}(L/\mathbb{Q})$, that there exist an infinite subset $J \subseteq \mathbb{V}(L)$ of unramified primes over $\mathbb{Q}$ and an element $\delta \in \mathrm{Gal}(L/\mathbb{Q})$ such that for each $\tilde{v} \in J$, if we write $v_{(-)} \stackrel{\text{def}}{=} \tilde{v}|_{(-)}$, then

- a [or, equivalently, any] Sylow-l subgroup of an inertia subgroup of G_1 associated to v_{K_1} maps isomorphically to an open subgroup of a Sylow-l subgroup of an inertia subgroup of G_2 associated to a prime $\in I$ via ϕ [so the decomposition subgroup of $\mathrm{Gal}(L/K_1)$ associated to $\tilde{v}$ maps into C via the surjection $\mathrm{Gal}(L/K_1) \twoheadrightarrow V \rtimes C$ — cf. Lemma 1.4], and
- $\mathrm{Fr}_{\tilde{v}/p_{\tilde{v}}} = \delta$.

Recall that since C and V acts on W_2, it is immediate that $\mathrm{Gal}(L/K_1)$ acts on W. Then we verify the following assertion:

Claim 1.7.A: δ^m acts trivially on W.

Indeed, we note that $\delta^n \in \mathrm{Gal}(L/K_1)$ maps to an element $c \in C$. Then, by considering the natural actions on [non-vanishing] Sylow-l subgroups of inertia subgroups, we observe that $c \in C = (\mathbb{Z}/l^a\mathbb{Z})^\times$ equals to $p_{\tilde{v}}^n$. On the other hand, since $n \mid (l-1) \mid m$, we conclude that $\delta^m \in \mathrm{Gal}(L/K_1)$ maps to the trivia element $\in C \subseteq V \rtimes C$, hence that δ^m acts trivially on W. This completes the proof of Claim 1.7.A.

Next, write $\Delta \subseteq \mathrm{Gal}(L/\mathbb{Q})$ for the subgroup generated by δ; $L^\Delta \subseteq L$ for the subfield fixed by Δ; $T \stackrel{\text{def}}{=} \mathrm{Res}_{L/L^\Delta} \circ \mathrm{Cor}_{L/L^\Delta} : \mathrm{Hom}(G_L, \mathbb{Q}_l) \to \mathrm{Hom}(G_L, \mathbb{Q}_l)$. Then we verify the following assertion:

Claim 1.7.B: $\dim_{\mathbb{Q}_l} T(W) = 1$.

Indeed, we note that for each $\tilde{v} \in J$, if we write $\phi(\tilde{v}) \in \mathbb{V}(E_a)$ for the prime [that lies over a prime $\in I$] determined by ϕ and $\tilde{v}$, then there exists a nonzero element $c_{\tilde{v}}$ of $\overline{\mathbb{Q}}_l$ such that

$$\mathrm{Cor}_{L/L^\Delta}(u)(\mathrm{Fr}_{v_{L^\Delta}}) \;=\; u(\mathrm{Fr}_{\tilde{v}}) \;=\; c_{\tilde{v}}u(\mathrm{Fr}_{\phi(\tilde{v})}) \;=\; c_{\tilde{v}}\mathrm{Cor}_{E_a/M_a}(u)(\mathrm{Fr}_{\phi(\tilde{v})|_{M_a}})$$

for each $u \in W_2$, where we also write u for the image via the isomorphism $W_2 \stackrel{\sim}{\to} W$. Then since there exist isomorphisms $(W \stackrel{\sim}{\to}) W_2 \stackrel{\sim}{\to} \mathbb{Q}_l(\zeta_{l^a})$ [that maps y_2 to 1], $C \stackrel{\sim}{\to} \mathrm{Gal}(\mathbb{Q}_l(\zeta_{l^a})/\mathbb{Q}_l)$ compatible with the respective natural actions, it follows immediately from Proposition 1.5 [cf. also the fact that J is infinite], together with the above displayed equality, that $\dim_{\mathbb{Q}_l} \mathrm{Cor}_{L/L^\Delta}(W) = 1$, hence that $\dim_{\mathbb{Q}_l} T(W) = 1$. This completes the proof of Claim 1.7.B.

Next, we verify the following assertion:

Claim 1.7.C: The natural action of δ on $\mathrm{Hom}(G_L, \mathbb{Q}_l)$ preserves W.

Indeed, write $D \subseteq \mathrm{Gal}(L/K_1)$ for the commutator subgroup of $\mathrm{Gal}(L/K_1)$. Then it is immediate that D is normal in $\mathrm{Gal}(L/\mathbb{Q})$, and the natural surjection $\mathrm{Gal}(L/K_1) \twoheadrightarrow V \rtimes C$ induces a surjection $D \twoheadrightarrow V$. Write $\overline{(-)} \stackrel{\text{def}}{=} (-) \otimes_{\mathbb{Q}_l} \overline{\mathbb{Q}}_l$. In light of the fact that D acts on $W \stackrel{\sim}{\to} W_2$ through $D \twoheadrightarrow V$, by considering the eigenvector space decomposition of the action on $\overline{W}_2$ of a generator of V, together with Claim 1.7.B, we obtain a common eigenvector $w \in \overline{W}$ with respect to the natural actions of elements of D such that $z \stackrel{\text{def}}{=} Tw \in \mathrm{Hom}(G_L, \mathbb{Q}_l)$ [not $\overline{\mathrm{Hom}(G_L, \mathbb{Q}_l)}$!] is nonzero. Write $Z \stackrel{\text{def}}{=} \mathbb{Q}_l[D]z \subseteq \mathrm{Hom}(G_L, \mathbb{Q}_l)$;

$\overline{W}^{\dagger} \stackrel{\text{def}}{=} \overline{\mathbb{Q}}_l[\Delta]w \subseteq \overline{\mathrm{Hom}(G_L, \mathbb{Q}_l)}$. Then since D is normal in $\mathrm{Gal}(L/\mathbb{Q})$, and $\delta(z) = z$, we observe that δ preserves Z. Moreover, it follows immediately from our choice of w, together with the definition of T, that $\overline{Z} \subseteq \overline{W}^{\dagger}$. Here, we note that since D is normal in $\mathrm{Gal}(L/\mathbb{Q})$, each $\delta^j w$ is a D-eigenvector whose character has order l^a. Therefore, the character decomposition of the D-module $\overline{W}^{\dagger}$ and the semisimplicity of $\overline{Z}$ as a D-module imply that there exists an element $d \in D$ such that one of the eigenvalues of d with respect to the action on $\overline{Z}$ is a primitive l^a-th root of unity. This immediately implies that $m \leq \dim_{\mathbb{Q}_l} Z$. On the other hand, since $\delta^m(w) = w$ [cf. Claim 1.7.A], it holds that $\dim_{\overline{\mathbb{Q}}_l} \overline{W}^{\dagger} \leq m$. Then it holds that $\overline{Z} = \overline{W}^{\dagger}$, hence that $w \in \overline{Z} \cap \overline{W} \neq \{0\}$. In particular, $Z \cap W \neq \{0\}$. Note that W is an irreducible D-module. Thus, we conclude that $W \subseteq Z$, hence from the fact that $\dim_{\mathbb{Q}_l} W = \dim_{\mathbb{Q}_l} Z = m$ that $Z = W$. This completes the proof of Claim 1.7.C.

Note that since the action of $V \rtimes C$ on W_2 is faithful, it follows from Claim 1.7.C that δ normalizes $\mathrm{Gal}(L/L_1)$. Then, in light of the structure of $V \rtimes C$, the conjugation by [a lifting of] δ preserves the pull-back of the mod l^a cyclotomic character via ϕ. Finally, for $i = 1, 2$, write $\chi_i : G_i \to \mathbb{Z}_l$ for the composite of the l-adic cyclotomic character $G_i \to \mathbb{Z}_l^{\times}$ with the l-adic logarithm $\mathbb{Z}_l^{\times} \to \mathbb{Z}_l$. Write $\rho_l \stackrel{\text{def}}{=} \chi_2 \circ \phi - \chi_1 : G_1 \to \mathbb{Z}_l$; $\rho_{l,a} : G_1 \to \mathbb{Z}/l^a\mathbb{Z}$ for the composite of ρ_l with the natural quotient. In light of our choice of L, our goal is to prove that ρ_l is trivial. In the above argument, since $\mathrm{Gal}(K_1/\mathbb{Q})$ is finite, by considering sufficiently large a, we may assume that the following condition holds:

> Let $g \in \mathrm{Gal}(K_1/\mathbb{Q})$. Then, if $\rho_{l,a} = \rho_{l,a} \circ g^{-1}(-)g$, then $\rho_l = \rho_l \circ g^{-1}(-)g$.

Observe that $\rho_{l,a} = \rho_{l,a} \circ \delta^{-1}(-)\delta$. Thus, $\rho_l = \rho_l \circ \delta^{-1}(-)\delta$. Write $K \stackrel{\text{def}}{=} K_1^{\Delta}$. Then since n and l is coprime, it is immediate that ρ_l extends to $\rho_l^{\dagger} : G_K \to \mathbb{Z}_l$. Again, we recall the local compatibility of l-adic cyclotomic characters at the primes v_1 where a Sylow-l subgroup of an inertia subgroup of G_1 associated to v_1 maps injectively to an open subgroup of a Sylow-l subgroup of an inertia subgroup of G_2. Then it follows immediately from Proposition 1.5, together with our choice of δ, that $\rho_l^{\dagger} = 0$, hence that $\rho_l = 0$. This completes the proof of Lemma 1.7. □

Lemma 1.8. *In the notation of Lemma 1.4, there exist positive integers m, n such that one may choose S_l satisfying $\#S_l \leq m$ for each $l \equiv 1 \pmod{n}$.*

Proof. First, we may assume that $\Omega_1 = \overline{\mathbb{Q}}$. In light of Lemma 1.7, let n be a positive integer such that for each sufficiently large prime number $l \equiv 1 \pmod{n}$, there exists a finite Galois extension $K_1 \subseteq M$ such that $\zeta_l \in M$, and the restriction of ϕ to G_M is compatible with the respective l-adic cyclotomic characters. For each such l, fix an S_l of minimal cardinality. Our goal is to verify that there exists a uniform bound m of $\#S_l$. Write $G_{1,\{l\}}^{l,M}$ for the quotient of G_1 that corresponds to the maximal pro-l extension of M unramified outside the primes over l; $G_{2,S_l}^{l,\zeta}$ for the quotient of G_2 that corresponds to the maximal pro-l extension of $K_2(\zeta_l)$ unramified outside the primes over S_l. Then it follows immediately from the definition of S_l that ϕ induces an open homomorphism $G_{1,\{l\}}^{l,M} \to G_{2,S_l}^{l,\zeta}$, hence the composite of natural injective homomorphisms

$$H^1(G_{2,S_l}^{l,\zeta}, \mathbb{Z}_l(1)) \hookrightarrow H^1(G_{1,\{l\}}^{l,M}, \mathbb{Z}_l(\rho)) \hookrightarrow H^1(G_{M,\{l\}}^{l}, \mathbb{Z}_l(1))^{\epsilon},$$

where $G_{M,\{l\}}^{l} \subseteq G_{1,\{l\}}^{l,M}$ denotes the open subgroup associated to $K_1 \subseteq M$; $\rho : G_{1,\{l\}}^{l,M} \to \mathbb{Z}_l^{\times}$ denotes a character whose restriction to $G_{M,\{l\}}^{l}$ coincides with the l-adic cyclotomic character; $\epsilon : \mathrm{Gal}(M/K_1) \to \mathbb{Z}_l^{\times}$ denotes the finite character obtained by forming the difference between ρ and the l-adic cyclotomic character. Thus, since $\mathrm{Gal}(M/K_1)$ acts transitively on the primes on M over a prime on K_1, the existence of a uniform bound m follows immediately from the Dirichlet's unit theorem, together with Kummer theory. [One may take m to be $2[K_1 : \mathbb{Q}]$.] This completes the proof of Lemma 1.8. □

Next, we prove a key observation on the finiteness of fibers [cf Proposition 1.10, (i)] and its application to a result concerning preservations of residue characteristic and Frobenius liftings [cf Proposition 1.10, (ii)]. The latter is crucial to prove the desired compatibility of cyclotomic characters. For the latter, we prove the following lemma, which may be regarded as a direct consequence of the Grunewald-Segal theorem.

Lemma 1.9. *Let K be a number field that is Galois over $\mathbb{Q}$. Write $\Sigma_K \subseteq \mathfrak{Primes}$ for the image of $\mathbb{V}(K)^{d=1}$ via the natural surjection $\eta : \mathbb{V}(K) \twoheadrightarrow \mathfrak{Primes}$. Let*

$$\Sigma_K \ = \ \mathcal{N} \coprod \left(\coprod_{1 \leq i \leq m} \Sigma_i \right)$$

be a finite partition such that the natural density of $\mathcal{N}$ is equal to 0. Then there exists a positive integer $i \leq m$ such that Σ_i is ample [cf. [6]].

Proof. Note that the respective natural densities of $\eta^{-1}(\mathcal{N})$, $\mathbb{V}(K)^{d=1}$ are equal to 0, 1. Thus, by applying [6], Theorem A, we conclude that there exists a positive integer $i \leq m$ such that $\eta^{-1}(\Sigma_i)$ is ample, which implies that Σ_i is ample. This completes the proof of Lemma 1.9. □

Remark 1.9.1. In the notation of Lemma 1.9, suppose that $K = \mathbb{Q}(\zeta_n)$ for some positive integer $n \geq 3$. Then $\Sigma_K = \{l \in \mathfrak{Primes} \mid l \equiv 1 \pmod{n}\}$.

Proposition 1.10. *In the notation of Definition 1.1, suppose that K_2 is totally imaginary. Then the following hold:*

(i) Let l be a prime number; $S_l \subseteq \mathbb{V}(K_2)$ a finite subset as in Lemma 1.4. For each $w \in \mathbb{V}(K_2) \setminus S_l$, write

$$V_w \ \subseteq \ \mathbb{V}(K_1)$$

for the subset of finite primes v on K_1 such that. for some prime number $l^\dagger \neq p_w$, a [or, equivalently, any] Sylow-$l^\dagger$ subgroup of an inertia subgroup of G_1 associated to v maps injectively to an open subgroup of a Sylow-$l^\dagger$ subgroup of an inertia subgroup of G_2 associated to w. Then there exists a finite subset $E \subseteq \mathbb{V}(K_2)$ independent of l such that for each $w \in \mathbb{V}(K_2)^{d=1} \setminus (S_l \cup E)$, it holds that V_w is finite.

(ii) Suppose that $K_2 \subseteq K_1$ $(\subseteq \overline{\mathbb{Q}})$, and K_1 is Galois over $\mathbb{Q}$. Then there exists a subset $T \subseteq \mathbb{V}(K_1)^{d=1}$ such that:

- *The subset $\eta(T) \subseteq \eta(\mathbb{V}(K_1)^{d=1})$ is cofinite, where $\eta : \mathbb{V}(K_1) \twoheadrightarrow \mathfrak{Primes}$ denotes the natural surjection.*
- *For each $v \in T$, $\phi(\mathrm{Fr}_v)$ is a lifting of the Frobenius associated to a finite prime $\in \mathbb{V}(K_2)^{d=1}$ that lies over p_v.*

Proof. First, we verify assertion (i). Fix a prime number p. Since K_2 is totally imaginary, it holds that $\mathrm{rank}_{\mathbb{Z}_p}\mathrm{Hom}_{\mathrm{cont}}(G_2, \mathbb{Z}_p) \geq 2$ [cf. [15], Corollary 10.3.7]. Write

$$f : \mathrm{Hom}_{\mathrm{cont}}(G_2, \mathbb{Z}_p) \ \overset{\phi^*}{\hookrightarrow} \ \mathrm{Hom}_{\mathrm{cont}}(G_1, \mathbb{Z}_p) \ = \ \mathrm{Hom}_{\mathrm{cont}}(G_{K_1}, \mathbb{Z}_p) \ \longrightarrow \ \mathrm{Hom}_{\mathrm{cont}}(G_{\mathbb{Q}}, \mathbb{Z}_p)$$

for the composite of the homomorphism obtained by pulling-back via ϕ with the corestriction $\mathrm{Cor}_{K_1/\mathbb{Q}}$. For each number field M, write $\chi_M : G_M \to \mathbb{Z}_p$ for the composite of the p-adic cyclotomic character $G_M \to \mathbb{Z}_p^\times$

with the p-adic logarithm. Here, we observe that it follows immediately from Lemma 1.4 that there exists an infinite subset $C \subseteq \mathbb{V}(K_1)$ such that $(\phi^*(\chi_{K_2}) - \chi_{K_1})(\mathrm{Fr}_v) = 0$ for any $v \in C$. Therefore, by applying Corollary 1.6, we conclude that $\mathrm{Cor}_{K_1/\mathbb{Q}}(\phi^*(\chi_{K_2}) - \chi_{K_1}) = 0$, hence that $f(\chi_{K_2}) = \mathrm{Cor}_{K_1/\mathbb{Q}}(\chi_{K_1}) = c\chi_{\mathbb{Q}}$ for some $c \in \mathbb{Q}_p$. In particular, it holds that $f(\chi_{K_2}) \neq 0$. Let $\psi \in \mathrm{Ker}(f) \setminus \{0\}$ [cf. our assumption that $\mathrm{rank}_{\mathbb{Z}_p}\mathrm{Hom}_{\mathrm{cont}}(G_2, \mathbb{Z}_p) \geq 2$]. Write

$$E \stackrel{\mathrm{def}}{=} \{w' \in \mathbb{V}(K_2)^{d=1} \mid \psi(\mathrm{Fr}_{w'}) = 0 \text{ if } w' \nmid p\}$$

[so E is independent of l]. Then it follows immediately from Proposition 1.5 that E is finite. Let $w \in \mathbb{V}(K_2)^{d=1} \setminus (S_l \cup E)$ [so $w \nmid p$, and $\psi(\mathrm{Fr}_w) \neq 0$]. Write $c_w \stackrel{\mathrm{def}}{=} \frac{\psi(\mathrm{Fr}_w)}{\chi_{K_2}(\mathrm{Fr}_w)} \in \mathbb{Q}_p^{\times}$. Then it holds that

$$0 \neq f(\psi - c_w\chi_{K_2}) = \mathrm{Cor}_{K_1/\mathbb{Q}}(\phi^*(\psi) - c_w\phi^*(\chi_{K_2})).$$

On the other hand, in light of the definition of c_w, it follows immediately from the well-known structure of a local decomposition group that $\psi - c_w\chi_{K_2}$ vanishes on [any conjugate of] the whole decomposition subgroup D_w that contains Fr_w. Recall that any decomposition subgroup of G_1 associated to an element $\in V_w$ maps into a conjugate of D_w [cf. Lemma 1.4]. Therefore, the homomorphism $\phi^*(\psi) - c_w\phi^*(\chi_{K_2})$ vanishes on $\{\mathrm{Fr}_v\}_{v \in V_w}$. Thus, we conclude from Corollary 1.6, together with the fact that $\mathrm{Cor}_{K_1/\mathbb{Q}}(\phi^*(\psi) - c_w\phi^*(\chi_{K_2})) \neq 0$, that V_w is finite. This completes the proof of assertion (i).

Next, we verify assertion (ii). In light of the existence of uniform bound m as in Lemma 1.8, it follows immediately from assertion (i), together with Lemma 1.9; Remark 1.9.1, that there exists a finite subset $S \subseteq \mathbb{V}(K_2)$ satisfying the following condition:

> Let $w \in \mathbb{V}(K_2)^{d=1} \setminus S$. Then there exist $v \in V_w$ and an ample subset $\Sigma_v \subseteq \mathfrak{Primes} \setminus \{p_v, p_w\}$ such that for each $l^{\dagger} \in \Sigma_v$, a [or, equivalently, any] Sylow-$l^{\dagger}$ subgroup of an inertia subgroup of G_1 associated to v maps injectively to an open subgroup of a Sylow-$l^{\dagger}$ subgroup of an inertia subgroup of G_2 associated to w.

Fix such an S. Write $\eta^{\dagger} : \mathbb{V}(K_2) \twoheadrightarrow \mathfrak{Primes}$ for the natural surjection; $T^{\dagger} \stackrel{\mathrm{def}}{=} \eta^{-1}(\eta(\mathbb{V}(K_1)^{d=1}) \setminus \eta^{\dagger}(S)) \subseteq \mathbb{V}(K_1)^{d=1}$. Let $w \in \mathbb{V}(K_2)$ be such that $p_w = \eta^{\dagger}(w) \in \eta(T^{\dagger})$ [so our assumption that $K_2 \subseteq K_1$, and K_1 is Galois over $\mathbb{Q}$ implies that $w \in \mathbb{V}(K_2)^{d=1} \setminus S$]; $v \in V_w$; $\Sigma_v \subseteq \mathfrak{Primes}$ as in the above condition. Write $\iota_v : G_{k_v} \to \mathrm{Aut}(\widehat{\mathbb{Z}}^{\Sigma_v}(1))$, $\iota_w : G_{k_w} \to \mathrm{Aut}(\widehat{\mathbb{Z}}^{\Sigma_v}(1))$ for the respective natural homomorphisms, where $k_{(-)}$ denotes the residue field at $(-)$. Then it follows immediately from Lemma 1.4 that $\mathrm{Im}(\iota_v) \subseteq \mathrm{Im}(\iota_w)$. Moreover, since Σ_v is ample, we *observe* from the definition of ampleness that:

- ι_v, ι_w are injective.
- $p_v = p_w$ [so $v \in T^{\dagger}$], and $\phi(\mathrm{Fr}_v)$ is a lifting of the Frobenius associated to w.

Write $T \subseteq T^{\dagger}$ for the subset of such "v"s obtained by varying w. Then it follows immediately from the various definitions involved that T satisfies the conditions in the statement of assertion (ii). This completes the proof of assertion (ii), hence of Proposition 1.10. □

Finally, we obtain our main result in the present section, which may be regarded as an affirmative resolution of Uchida's conjecture:

Theorem 1.11. *In the notation of Definition 1.1, the open homomorphism ϕ arises from a uniquely determined commutative diagram*

$$\begin{array}{ccc} \Omega_2 & \longrightarrow & \Omega_1 \\ \uparrow & & \uparrow \\ K_2 & \longrightarrow & K_1 \end{array}$$

of field embeddings, where the vertical arrows denote the natural inclusions.

Proof. To verify Theorem 1.11, by a formal Galois descent argument, we may assume without loss of generality that $\Omega_1 = \overline{\mathbb{Q}}$, K_2 is totally imaginary, $K_2 \subseteq K_1$, and K_1 is Galois over $\mathbb{Q}$. Write

$$\theta \stackrel{\text{def}}{=} \chi_1^{-1} \cdot \phi^* \chi_2 : G_1 \longrightarrow \mathrm{Aut}(\widehat{\mathbb{Z}}(1)),$$

where $\chi_{(-)} : G_{(-)} \to \mathrm{Aut}(\widehat{\mathbb{Z}}(1))$ denotes the cyclotomic character; $H \stackrel{\text{def}}{=} \mathrm{Ker}(\theta)$. Then it follows immediately from Chebotarev's density theorem, together with Proposition 1.10, (ii), that

$$G_1 = \overline{\bigcup_{\sigma \in \mathrm{Gal}(K_1/\mathbb{Q})} \sigma^{-1} H \sigma},$$

where $\overline{(-)}$ denotes the closure of $(-)$. This implies that $H \subseteq G_1$ is an open subgroup. Then, in light of a formal Galois descent argument again, by replacing G_1 by H, we may assume that ϕ is compatible with the respective cyclotomic characters. Thus, by applying Theorem 1.2, we obtain the desired conclusion. This completes the proof of Theorem 1.11. □

Remark 1.11.1. It may be natural to ask whether or not the positive characteristic analogue of Theorem 1.11 holds. Indeed, some partial results are already discussed by M. Saïdi and A. Tamagawa [cf. [20]]. However, our discussion in the present section does not seem to work in the positive characteristic setting. In fact, the above discussion heavily depends on Proposition 1.5 [whose proof is based on the rank argument] and the existence of absolute constant field $\mathbb{Q}$. Thus, at the time of writing of the present paper, the author has no clue to prove an unconditional generalization of the Saïdi-Tamagawa's result.

2 Semi-absoluteness of open homomorphisms

In the present section, we establish the semi-absoluteness of open homomorphisms between the absolute Galois groups of finitely generated fields in characteristic 0. First, we observe that the analogue for smooth varieties over finitely generated fields of characteristic 0 is much easier to prove.

Lemma 2.1. *Let K be a Hilbertian field. Then there exists no nontrivial topologically finitely generated normal closed subgroup of an open subgroup of G_K.*

Proof. Since any finite extension field of a Hilbertian field is again Hilbertian [cf. [4], Proposition 13.3.3], Lemma 2.1 follows immediately from Weissauer's theorem [cf. [4], Theorem 15.3.1, (b)], together with [4], Theorem 19.2.2. □

Remark 2.1.1. Note that any finitely generated field of characteristic 0 is Hilbertian [cf., e.g., [4], Theorem 14.4.2].

Proposition 2.2. *Let K be a field of characteristic 0; L a Hilbertian field; X a smooth variety over K;*

$$\phi : \Pi_X \longrightarrow G_L$$

an open homomorphism. Then $\phi(\Delta_X) = \{1\}$.

Proof. Note that Δ_X is a topologically finitely generated normal closed subgroup of Π_X [cf. [8]; [9]; [14], Proposition 2.2]. Thus, Proposition 2.2 follows immediately from Lemma 2.1. □

Next, we prove the function field analogue of Proposition 2.2 in the case where the base fields are number fields.

Lemma 2.3. *Let p be a prime number; K a number field; $S \subseteq \mathfrak{Primes}$ a subset that contains p. For each field extension $K \subseteq L$ contained in K_{p^∞}, write $G^p_{L,S}$ for the quotient of G_L that corresponds to the maximal pro-p extension of L unramified outside the primes over S. Then $G^p_{K_{p^\infty},S}$ is not topologically finitely generated.*

Proof. Note that, for each positive integer $m \geq 2$, the field $K(\zeta_{p^m})$ is totally imaginary, and the cardinality of complex places of $K(\zeta_{p^m})$ is $\geq \frac{p^m - p^{m-1}}{2}$. Therefore, it follows from [15], Corollary 10.3.7, that the rank of pro-p group $G^p_{K(\zeta_{p^m}),S}$ is $\geq \frac{p^m - p^{m-1}}{2}$. Thus, in light of the natural isomorphism $G^p_{K_{p^\infty},S} \stackrel{\sim}{\to} \varprojlim_m G^p_{K(\zeta_{p^m}),S}$ of pro-p groups, since the rank of the pro-p group $\mathrm{Gal}(K_{p^\infty}/K(\zeta_{p^m}))$ is 1 [where $m \geq 2$], we conclude that $G^p_{K_{p^\infty},S}$ is not topologically finitely generated. This completes the proof of Lemma 2.3. $\square$

Lemma 2.4. *Let l be a prime number; K a finitely generated extension field of $\mathbb{Q}^{\mathrm{ab}}$ [i.e., the maximal abelian extension field of $\mathbb{Q}$]; X a proper, normal variety over K. Then there exists no nontrivial G_K-equivariant homomorphism $\Delta_X^{\mathrm{ab}} \to \mathbb{Z}_l$.*

Proof. First, since $\mathbb{Z}_l$ is torsion-free, it follows immediately from [14], Proposition A.6, (iv), that, by replacing X by the Albanese variety of X, we may assume without loss of generality that X is an abelian variety. Moreover, in light of a suitable specialization argument, we may assume without loss of generality that K is a finite extension field of $\mathbb{Q}^{\mathrm{ab}}$. In this situation, the nonexistence assertion follows immediately from [11], Appendix, Theorem 1, together with the duality theory of abelian varieties. This completes the proof of Lemma 2.4. $\square$

Definition 2.5. Let K be a field of characteristic 0; X a smooth curve [i.e., a smooth and 1-dimensional algebraic variety] over K. Write $\overline{X}$ for the smooth compactification of X over K. Then:

(i) We shall refer to an element of $\overline{X}(\overline{K}) \setminus X(\overline{K})$ as a *cusp* of X.

(ii) We shall say that X is a *hyperbolic curve* if $2g - 2 + r > 0$, where g denotes the genus of X; r denotes the cardinality of the set of cusps of X.

(iii) Suppose that X is a hyperbolic curve over K. Then we shall refer to the stabilizer subgroup of Δ_X associated to some pro-cusp of the pro-universal étale covering of $X_{\overline{K}}$ that lies over a cusp $x \in \overline{X}(\overline{K}) \setminus X(\overline{K})$ as a *cuspidal inertia subgroup* of Π_X or Δ_X associated to x. Note that, for each cuspidal inertia subgroup $I \subseteq \Delta_X$, it holds that $I \cong \widehat{\mathbb{Z}}(1)$ [where "(1)" denotes the Tate twist], and $I = N_{\Delta_X}(I)$ [cf., e.g., [13], Proposition 1.2, (ii)].

(iv) Write $\{U_i\}_{i \in I}$ for the family of open subschemes of $X_{\overline{K}}$ that are hyperbolic curves over $\overline{K}$. Note that there exists a natural isomorphism of profinite groups

$$\Delta_{K(X)} \stackrel{\sim}{\to} \varprojlim_{i \in I} \Delta_{U_i},$$

where the transition maps are the [outer] surjections induced by the natural open immersions. Then we shall refer to an inverse limit of cuspidal inertia subgroups of some cofinal collection of Δ_{U_i}'s as a *cuspidal inertia subgroup* of $G_{K(X)}$ or $\Delta_{K(X)}$. It follows immediately from the corresponding properties in the case of hyperbolic curves mentioned above that, for each cuspidal inertia subgroup $I \subseteq \Delta_{K(X)}$, it holds that $I \cong \widehat{\mathbb{Z}}(1)$, and $I = N_{\Delta_{K(X)}}(I)$.

Proposition 2.6. *Let K be a finitely generated field of characteristic 0; L a number field; X a smooth, proper curve over K;*

$$\phi : G_{K(X)} \longrightarrow G_L$$

an open homomorphism. Then $\phi(\Delta_{K(X)}) = \{1\}$.

Proof. First, in light of Lemma 2.1, together with Remark 2.1.1, to verify that $\phi(\Delta_{K(X)}) = \{1\}$, by replacing L by a finite extension field of L, we may assume without loss of generality that $\zeta_4 \in L$, hence that L is totally imaginary. In particular, the profinite group G_L has no torsion element. On the other hand, since Δ_X is topologically finitely generated, it follows immediately from Lemma 2.1, together with Remark 2.1.1, again, that, to verify that $\phi(\Delta_{K(X)}) = \{1\}$, it suffices to verify that any cuspidal inertia subgroup of $\Delta_{K(X)}$ vanishes via ϕ.

Suppose that there exists a cuspidal inertia subgroup $I \subseteq \Delta_{K(X)}$ such that $\phi(I) \neq \{1\}$. Then since G_L has no torsion element, it follows immediately from the structure of a cuspidal inertia subgroup that there exists a prime number p such that ϕ induces an isomorphism $I_p \xrightarrow{\sim} \phi(I_p)$, where $(\mathbb{Z}_p(1) \xrightarrow{\sim}) I_p \subseteq I (\xrightarrow{\sim} \widehat{\mathbb{Z}}(1))$ denotes the p-part of I. To derive a contradiction, since the extension degree $[L(\zeta_p) : L]$ is prime to p, by replacing L by $L(\zeta_p)$, if necessary, we may assume without loss of generality that

$$\zeta_p \in L.$$

Next, we observe that the natural surjective homomorphism $G_{K(X)} \twoheadrightarrow G_K$ induces a natural splitting exact sequence

$$1 \longrightarrow I \longrightarrow N_{G_{K(X)}}(I) \longrightarrow G_{K'} \longrightarrow 1,$$

where K' $(\subseteq \overline{K})$ denotes a finite extension field of K [cf. the final sentence of Definition 2.5, (iv)]. Fix a splitting s of the above exact sequence. Let $J_p \subseteq s(G'_K)$ be a closed subgroup isomorphic to $\mathbb{Z}_p$ such that the natural conjugation action of J_p on I_p is faithful. In particular, since $I_p \xrightarrow{\sim} \phi(I_p)$, the open homomorphism ϕ induces an isomorphism $I_p \rtimes J_p \xrightarrow{\sim} \phi(I_p) \rtimes \phi(J_p)$ $(\subseteq G_L)$. Then, in light of the structure of $I_p \rtimes J_p$, it follows immediately from [3], Theorem 5.4.3, that $\phi(I_p) \rtimes \phi(J_p)$ is contained in a decomposition subgroup $D_l \subseteq G_L$ associated to a prime on L whose residue characteristic $l \neq p$. Moreover, in light of the structure of D_l, one may observe that $\phi(I_p)$ is contained in the inertia subgroup of D_l. Here, we note that the above discussion implies that:

> Claim 2.6.A: For any cuspidal inertia subgroup $I' \subseteq \Delta_{K(X)}$, the image via ϕ of the p-portion of I' is contained in an inertia subgroup of G_L associated to a prime on L whose residue characteristic $\neq p$.

Next, write

$$H \stackrel{\text{def}}{=} s(G_{K'_{p^\infty}}).$$

Then since $\phi(H)$ normalizes $\phi(I_p)$ $(\subseteq D_l)$, it follows immediately from [15], Corollary 12.1.3, that $\phi(H) \subseteq D_l$. Moreover, since $\phi(H)$ commutes with $\phi(I_p)$, it follows immediately from the well-known structure of the tame quotient of D_l that:

> Claim 2.6.B: $\phi(H)$ vanishes in the pro-p unramified quotient $D_l \twoheadrightarrow \mathbb{Z}_p$.

Next, we verify the following assertion:

> Claim 2.6.C: The composite
>
> $$G_{K(X)} \xrightarrow{\phi} G_L \xrightarrow{\chi_p} \mathbb{Z}_p \ (\subseteq \mathbb{Z}_p^\times)$$
>
> [cf. our assumption that $\zeta_p \in L$] factors through the natural quotient $G_{K(X)} \twoheadrightarrow G_K$.

Indeed, since any closed subgroup of $\mathbb{Z}_p$ is isomorphic to $\mathbb{Z}_p$, it suffices to verify that any continuous surjective homomorphism $\psi : G_{K(X)} \twoheadrightarrow \mathbb{Z}_p$ factors through the natural quotient $G_{K(X)} \twoheadrightarrow G_K$. Note that ψ induces a G_K-equivariant homomorphism $\Delta_{K(X)}^{\mathrm{ab}} \to \mathbb{Z}_p$ [where we regard $\mathbb{Z}_p$ as a G_K-module via the trivial action]. Then since the p-adic cyclotomic character associated to G_K is open, it follows immediately from Lemma 2.4, together with the well-known structure of $\Delta_{K(X)}^{\mathrm{ab}}$, that $\psi(\Delta_{K(X)}) = \{1\}$, as desired. This completes the proof of Claim 2.6.C.

Next, in light of Claim 2.6.C, we observe that there exists a $\mathbb{Z}_p$-extension $K \subseteq M$ such that ϕ induces a continuous homomorphism

$$\psi_0 : \Delta_{K(X)} \overset{\mathrm{out}}{\rtimes} G_{(K' \cdot M)_{p^\infty}} \longrightarrow G^p_{L_{p^\infty}}$$

whose cokernel is topologically finitely generated. On the other hand, it follows immediately from Claims 2.6.A; 2.6.B, together with the various definitions involved, that the image of $\mathrm{Im}(\psi_0)$ in the quotient $G^p_{L_{p^\infty}} \twoheadrightarrow G^p_{L_{p^\infty},\{p\}}$ [cf. the notation of Lemma 2.3] is topologically finitely generated. Then since the cokernel of ψ_0 is topologically finitely generated, it holds that $G^p_{L_{p^\infty},\{p\}}$ is topologically finitely generated. However, this contradicts Lemma 2.3. Thus, we conclude that any cuspidal inertia subgroup of $\Delta_{K(X)}$ vanishes via ϕ, as desired. This completes the proof of Proposition 2.6. □

Theorem 2.7. *For $i = 1, 2$, let k_i be a number field; K_i a finitely generated regular extension field of k_i;*

$$\phi : G_{K_1} \longrightarrow G_{K_2}$$

an open homomorphism. Then ϕ induces an open homomorphism

$$G_{k_1} \longrightarrow G_{k_2}$$

compatible with the natural surjective homomorphisms $G_{K_1} \twoheadrightarrow G_{k_1}$, $G_{K_2} \twoheadrightarrow G_{k_2}$.

Proof. First, it follows immediately from the various definitions involved that we may assume without loss of generality that $k_2 = K_2$. Next, we note that, if the transcendental degree of K_1 over k_1 is a positive integer m, then one may regard K_1 as the function field of a smooth, proper curve over a finitely generated extension field of k_1 of transcendental degree $m - 1$. Thus, in light of induction on the transcendental degree of K_1 over k_1, we conclude from Proposition 2.6 that ϕ factors through the natural surjective homomorphism $G_{K_1} \twoheadrightarrow G_{k_1}$. This completes the proof of Theorem 2.7. □

3 Higher dimensional generalization of Uchida's conjecture

In the present section, by combining the affirmative answer to the Uchida's conjecture, the semi-absoluteness of certain open homomorphisms obtained in the previous section, with a highly nontrivial result in relative anabelian geometry obtained by Mochizuki, we prove a higher dimensional analogue of the Uchida's conjecture for finitely generated fields of characteristic 0.

First, we recall Mochizuki's famous results in relative anabelian geometry.

Definition 3.1. Let p be a prime number; F a field. Then we shall say that F is *sub-p-adic* if F is isomorphic to a subfield of a finitely generated extension field of $\mathbb{Q}_p$. In particular, any finitely generated field of characteristic 0 is sub-p-adic.

Definition 3.2. Let K be a field. Then:

(i) Let X, Y be algebraic varieties over K. Then we shall write

$$\mathrm{Mor}_K^D(X, Y)$$

for the set of dominant morphisms $X \to Y$ over K;

$$\mathrm{Hom}_{G_K}^{\mathrm{open}}(\Pi_X, \Pi_Y)/\mathrm{Inn}(\Delta_Y)$$

for the set of open homomorphisms $\Pi_X \to \Pi_Y$ that lies over the identity automorphism of G_K, considered up to compositions with inner automorphism of Π_Y determined by elements of Δ_Y.

(ii) Let K_1, K_2 be finitely generated extension fields over K. Then we shall write

$$\mathrm{Emb}_K(K_2, K_1)$$

for the set of embeddings $K_2 \hookrightarrow K_1$ of fields over K;

$$\mathrm{Hom}_{G_K}^{\mathrm{open}}(G_{K_1}, G_{K_2})/\mathrm{Inn}(\Delta_{K_2})$$

for the set of open homomorphisms $G_{K_1} \to G_{K_2}$ that lies over the identity automorphism of G_K, considered up to compositions with inner automorphism of G_{K_2} determined by elements of $\Delta_{K_2} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(G_{K_2} \to G_K)$.

Theorem 3.3. *Let p be a prime number; K a sub-p-adic field. Then the following hold:*

(i) Let X be a smooth variety over K; Y a hyperbolic curve over K. Then the natural map

$$\mathrm{Mor}_K^D(X, Y) \longrightarrow \mathrm{Hom}_{G_K}^{\mathrm{open}}(\Pi_X, \Pi_Y)/\mathrm{Inn}(\Delta_Y)$$

is bijective.

(ii) Let X, Y be algebraic varieties over K. Then the natural map

$$\mathrm{Emb}_K(K(Y), K(X)) \longrightarrow \mathrm{Hom}_{G_K}^{\mathrm{open}}(G_{K(X)}, G_{K(Y)})/\mathrm{Inn}(\Delta_{K(Y)})$$

is bijective.

Proof. Theorem 3.3 follows immediately from [12], Theorems A, B. □

Definition 3.4.

(i) Let X, Y be integral schemes. Then we shall write

$$\mathrm{Mor}^D(X, Y)$$

for the set of dominant morphisms $X \to Y$;

$$\mathrm{Hom}^{\mathrm{open}}(\Pi_X, \Pi_Y)/\mathrm{Inn}(\Pi_Y)$$

for the set of open homomorphisms $\Pi_X \to \Pi_Y$, considered up to compositions with inner automorphism of Π_Y.

(ii) Let K_1, K_2 be fields. Then we shall write

$$\mathrm{Emb}(K_2, K_1)$$

for the set of embeddings $K_2 \hookrightarrow K_1$ of fields;

$$\mathrm{Hom}^{\mathrm{open}}(G_{K_1}, G_{K_2})/\mathrm{Inn}(G_{K_2})$$

for the set of open homomorphisms $G_{K_1} \to G_{K_2}$, considered up to compositions with inner automorphisms of G_{K_2}.

Finally, we obtain our main results, i.e., the geometricity of absolute open homomorphisms:

Theorem 3.5. *Let K_1, K_2 be finitely generated fields of characteristic 0. Then the natural map*

$$\psi : \mathrm{Emb}(K_2, K_1) \longrightarrow \mathrm{Hom}^{\mathrm{open}}(G_{K_1}, G_{K_2})/\mathrm{Inn}(G_{K_2})$$

is bijective.

Proof. First, we verify the injectivity of ψ. For $i = 1, 2$, write $k_i \subseteq K_i$ for the integral closure of the prime field in K_i, which is a number field. Note that any field embedding $K_2 \hookrightarrow K_1$ induces a field embedding $k_2 \hookrightarrow k_1$. Therefore, in light of the center-freeness of G_{k_2} [cf. [15], Corollary 12.1.6], the injectivity of ψ follows immediately from the injectivity portion of Theorems 1.11; 3.3, (ii). On the other hand, the surjectivity of ψ follows immediately from Theorem 2.7 and the surjectivity portions of Theorems 1.11; 3.3, (ii). This completes the proof of Theorem 3.5. □

Remark 3.5.1. Theorem 3.5 may be regarded as

- the affirmative result for the higher dimensional analogue of Uchida's conjecture for number fields and
- the Hom-version [in characteristic 0] of Pop's famous result concerning the Isom-version [cf., e.g., [17], [18]].

Remark 3.5.2. It may be natural to ask whether or not the positive characteristic analogue of Theorem 3.5 holds. However, as discussed in Remark 1.11.1, there is still no clue to prove an unconditional result. Moreover, due to the lack of a result concerning relative Hom-form of anabelian Grothendieck conjecture in positive characteristic, it appears to the author that it is also hopeless to obtain a higher dimensional positive characteristic analogue of Theorem 3.5 at the time of writing of the present paper.

Remark 3.5.3. Let $\Sigma \subseteq \mathfrak{Primes}$ be a nonempty subset. Then one may prove the geometrically pro-Σ version of Theorem 3.5. Indeed, in the case where $\Sigma \neq \mathfrak{Primes}$, the geometrically pro-$\Sigma$ analogue of Theorem 2.7 follows immediately from the fact that the absolute Galois group of a number field admits no nontrivial pro-Σ normal closed subgroup, and the geometrically pro-Σ analogue of the relative anabelian result due to Mochizuki follows immediately from the corresponding result due to Corry and Pop.

Remark 3.5.4. Let $\mathcal{C}$ be a set of fields. Then one may ask whether or not the analogous result holds for the function fields [of sufficiently large transcendental degree] over fields that belong to $\mathcal{C}$. Note that Theorem 3.5 may be regarded as the affirmative result in the case where $\mathcal{C}$ consists of number fields. On the other hand, if $\mathcal{C}$ contains an algebraically closed field, then it is difficult to expect that such an affirmative result exists. Indeed, it is known that the absolute Galois groups of 1-dimensional function fields are free profinite groups [cf. [7], [19]]. Thus, one may construct a continuous surjective homomorphism from the absolute Galois group of a 1-dimensional function field over an algebraically closed field to the absolute Galois group of a higher dimensional ≥ 2 function field over the same base field, which does not arise from a *natural* field embedding. [If the base field is an algebraic closure of the prime field, then this observation already gives a counter-example.] On the other hand, in [22], the author proved that the Isom-version holds for finitely generated transcendental fields over subfields of the maximal Kummer extensions of number fields. Thus, one may ask whether or not its open Hom-version holds. However, at the time of writing of the present paper, the author has no clue to obtain an affirmative result. [Note that one may not follow the strategy of the present paper due to the Shafarevich conjecture for the absolute Galois group of the maximal cyclotomic extension of a number field.]

Corollary 3.6. *Any open homomorphism between the absolute Galois groups of finitely generated fields of characteristic* 0 *with same transcendental degree is injective. In particular, the absolute Galois groups of finitely generated fields of characteristic* 0 *satisfy the Hopfian property.*

Proof. Corollary 3.6 follows immediately from Theorem 3.5. □

Remark 3.6.1. Note that Corollary 3.6 gives an affirmative answer to the question posed in [15], Question 12.3.3. Here, one may not drop the assumption that the transcendental degrees of fields under consideration are same. Indeed, if K is a field, then the natural surjective homomorphism $G_{K(t)} \twoheadrightarrow G_K$ is not injective, where t denotes an indeterminate.

Theorem 3.7. *Let X be a smooth variety of positive dimension over a number field; Y a hyperbolic curve over a finitely generated field of characteristic* 0*. Then the natural map*

$$\psi : \mathrm{Mor}^{D}(X, Y) \longrightarrow \mathrm{Hom}^{\mathrm{open}}(\Pi_X, \Pi_Y)/\mathrm{Inn}(\Pi_Y)$$

is bijective.

Proof. First, we verify the injectivity of ψ. Write k_X, k_Y for the respective base fields of X, Y. Note that any dominant morphism $X \to Y$ induces a field embedding $k_Y \hookrightarrow k_X$ [cf. [21], Lemma 4.2]. Therefore, in light of the center-freeness of G_{k_Y}, the injectivity of ψ follows immediately from the injectivity portions of Theorems 3.5; 3.3, (i). On the other hand, the surjectivity of ψ follows immediately from Proposition 2.2 and the surjectivity portions of Theorems 3.5; 3.3, (i). This completes the proof of Theorem 3.7. □

Remark 3.7.1. Let $\Sigma \subseteq \mathfrak{Primes}$ be a nonempty subset. Then one may prove the geometrically pro-Σ version of Theorem 3.7. Indeed, the geometrically pro-Σ analogue of Proposition 2.2 may be proved by a totally similar way to the proof of Proposition 2.2, and the Mochizuki's anabelian result applied in the proof of Theorem 3.7 holds in the geometrically pro-Σ setting.

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References

[1] S. Corry and F. Pop, Pro-p hom-form of the birational anabelian conjecture over sub-p-adic fields, *J. reine angew. Math.* **628** (2009), pp. 121–128.

[2] M. Emsalem, Sur les ideaux dont l'image par l'application d'Artin dans une $\mathbb{Z}_p$-extensions est triviale, *J. reine angew. Math.* **382** (1987), pp. 181–198.

[3] A. Engler and A. Prestel, *Valued Fields*, *Springer Monographs in Mathematics*, Springer-Verlag (2005).

[4] M. Fried and M. Jarden, *Field arithmetic (Fourth edition)*, *Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge, A Series of Modern Surveys in Mathematics* **11**, Springer-Verlag (2005).

[5] A. Grothendieck and M. Raynaud, *Revêtements étales et groupe fondamental* (SGA1), *Lecture Notes in Math.* **224** (1971), Springer-Verlag.

[6] F. J. Grunewald and D. Segal, On congruence topologies in number fields, *J. Reine Angew. Math.*, **311/312** (1979), pp. 389–396.

[7] D. Harbater, Fundamental groups and embedding problems in characteristic p. In: "*Recent Developments in the Inverse Galois Problem*" (M. Fried, et al., eds.), *AMS Contemp. Math. Series*, **186** (1995), pp. 353–369.

[8] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero. I, *Ann. Math.* **79** (1964), pp. 109–203.

[9] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero. II, *Ann. Math.* **79** (1964), pp. 205–326.

[10] Y. Hoshi, Homomorphisms of global solvably closed Galois groups compatible with cyclotomic characters, *Tohoku Math. J.* **77** (2025), pp. 17–32.

[11] N. Katz and S. Lang, Finiteness theorems in geometric class field theory, with an appendix by Kenneth A. Ribet, *Enseign. Math.* (2) **27** (1981), pp. 285–319.

[12] S. Mochizuki, The local pro-p anabelian geometry of curves, *Invent. Math.* **138** (1999), pp. 319–423.

[13] S. Mochizuki, A combinatorial version of the Grothendieck conjecture, *Tohoku Math. J.* **59** (2007), pp. 455–479.

[14] S. Mochizuki, Topics in absolute anabelian geometry I: Generalities, *J. Math. Sci. Univ. Tokyo* **19** (2012), pp. 139–242.

[15] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields*, *Grundlehren der Mathematischen Wissenschaften* **323**, Springer-Verlag (2000).

[16] OpenAI, Open homomorphisms of global solvably closed Galois groups, Manuscript, dated October 5, 2026.

[17] F. Pop, On Grothendieck's conjecture of birational anabelian geometry, *Ann. of Math.* **138** (1994), pp. 147–185.

[18] F. Pop, Étale Galois covers of affine smooth curves, *Invent. Math.* **120** (1995), pp. 555–578.

[19] F. Pop, Alterations and birational anabelian geometry, in H. Hauser, J. Lipman, F. Oort, A. Quirós, *Resolution of Singularities*, *Progress in Math.* **181** (2000), pp. 519–533, Birkhäuser-Verlag, Basel.

[20] M. Saïdi and A. Tamagawa, On the hom-form of Grothendieck's birational anabelian conjecture in characteristic $p > 0$, *Algebra and Number theory* **5** (2011), pp. 131–184.

[21] A. Tamagawa, The Grothendieck conjecture for affine curves, *Compos. Math.* **109** (1997), pp. 135–194.

[22] S. Tsujimura, Generalizations of birational anabelian Grothendieck conjecture for curves I, *RIMS Preprint* **1997** (July 2026).

[23] K. Uchida, Homomorphisms of Galois groups of solvably closed Galois extensions, *J. Math. Soc. Japan* **33** (1981), pp. 595–604.

[24] M. Waldschmidt, Transcendance et exponentielles en plusieurs variables, *Invent. Math.* **63** (1981), pp. 97–128.

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